

THE SPACE OF LINEAR ANTI-SYMPLECTIC INVOLUTIONS IS A HOMOGENOUS SPACE

PETER ALBERS AND URS FRAUENFELDER

ABSTRACT. In this note we prove that the space of linear anti-symplectic involutions is the homogenous space $\mathrm{Gl}(n, \mathbb{R}) \backslash \mathrm{Sp}(n)$. This result is motivated by the study of symmetric periodic orbits in the restricted 3-body problem.

1. INTRODUCTION

We denote by $(\mathbb{R}^{2n}, \omega)$ the standard symplectic vector space. Then

$$\mathrm{Sp}(n) := \{\psi \in \mathrm{Gl}(2n, \mathbb{R}) \mid \omega(\psi v, \psi w) = \omega(v, w), \forall v, w \in \mathbb{R}^{2n}\} \quad (1)$$

is the linear symplectic group. We denote by $\mathcal{A}(n)$ the space of linear anti-symplectic involutions

$$\mathcal{A}(n) := \{S \in \mathrm{Gl}(2n, \mathbb{R}) \mid S^2 = \mathbb{1} \text{ and } \omega(Sv, Sw) = -\omega(v, w), \forall v, w \in \mathbb{R}^{2n}\} \quad (2)$$

and by

$$R := \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \in \mathcal{A}(n) \quad (3)$$

the standard anti-symplectic involution written in matrix form with respect to the Lagrangian splitting $\mathbb{R}^{2n} \cong \mathbb{R}^n \oplus \mathbb{R}^n \cong T^*\mathbb{R}^n$. Finally, we abbreviate

$$\mathrm{Sp}^R(n) := \{\psi \in \mathrm{Sp}(n) \mid \psi = R\psi^{-1}R\}. \quad (4)$$

The group $\mathrm{Gl}(n, \mathbb{R})$ is identified with the subgroup

$$\left\{ \begin{pmatrix} A & 0 \\ 0 & (A^T)^{-1} \end{pmatrix} \mid A \in \mathrm{Gl}(n, \mathbb{R}) \right\} \quad (5)$$

of $\mathrm{Sp}(n)$. In this note we prove the following Theorems.

Theorem 1. $\mathrm{Sp}^R(n)$, $\mathcal{A}(n)$, and the homogeneous space $\mathrm{Gl}(n, \mathbb{R}) \backslash \mathrm{Sp}(n)$ are diffeomorphic.

Theorem 2. Every element in $\mathrm{Sp}(n)$ is conjugate in $\mathrm{Sp}(n)$ to an element in $\mathrm{Sp}^R(n)$, i.e. for all $\phi \in \mathrm{Sp}(n)$ there exists $\psi \in \mathrm{Sp}(n)$ with $\psi\phi\psi^{-1} \in \mathrm{Sp}^R(n)$.

Remark 3. The principal interest in the space $\mathrm{Sp}^R(n)$ comes from studying symmetric periodic orbits of Hamiltonian dynamical systems invariant under anti-symplectic involution. Indeed, the linearized Poincaré return map takes values in $\mathrm{Sp}^R(n)$. In the restricted 3-body problem an antisymplectic involution plays a crucial role already in the work of Birkhoff [Bir15]. In fact, the term *symmetric periodic orbit* originates from Birkhoff's work. For more details we refer the reader to [FvK12].

As in symplectic field theory it is important to understand the dichotomy of good and bad periodic orbits, cp. [EGH00]. This property only depends on the conjugacy class of the linearized Poincaré return map. In particular, by Theorem 2 the property of being *symmetric* does not pose any obstructions on the conjugacy class of the linearized Poincaré return map.

Remark 4. The homogeneous space structure is not unique. At the end of this article we give the Lagrangian Grassmannian a second homogenous structure, see Corollary 9.

2. PROOF OF THEOREMS 1 AND 2

As preparation we need

Lemma 5. $\mathrm{Gl}(n, \mathbb{R})$ equals

$$\{\psi \in \mathrm{Sp}(n) \mid R\psi = \psi R\} . \quad (6)$$

PROOF. That $\mathrm{Gl}(n, \mathbb{R})$ is contained in the given set follows from equation (5). For the opposite inclusion we recall that a matrix

$$\psi := \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad (7)$$

is in $\mathrm{Sp}(n)$ if and only if

$$AD^T - C^T B = \mathbb{1}, \quad A^T C = C^T A, \quad \text{and} \quad B^T D = D^T B , \quad (8)$$

see [MS98, Exercise 1.13]. Thus, the equality

$$R\psi R = \begin{pmatrix} A & -B \\ -C & D \end{pmatrix} \stackrel{!}{=} \psi \quad (9)$$

implies that $B = C = 0$ and since ψ is symplectic we conclude from $AD^T - C^T B = \mathbb{1}$ that $D^T = A^{-1}$. \square

The following Lemma is well-known. For the readers convenience we include a proof.

Lemma 6. Let $\mathbb{R}^{2n} = L_1 \oplus L_2$ be a Lagrangian splitting of \mathbb{R}^{2n} , i.e. L_1, L_2 are Lagrangian subspaces, then there exists bases v_1, \dots, v_n of L_1 and w_1, \dots, w_n of L_2 such that

$$\omega(v_i, w_j) = \delta_{ij}, \quad i, j = 1, \dots, n \quad (10)$$

that is, $v_1, \dots, v_n, w_1, \dots, w_n$ is a symplectic basis of \mathbb{R}^{2n} .

PROOF. We denote by $\Pi_1 : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ the projection to L_1 along L_2 and by $\Pi_2 = \mathbb{1} - \Pi_1$ the projection to L_2 along L_1 . We prove the Lemma by induction. For $1 \leq k \leq n$ we denote by **A(k)** the following assertion.

A(k): There exist linearly independent vectors $v_1, \dots, v_k \in L_1$ and $w_1, \dots, w_k \in L_2$ such that

$$\omega(v_i, w_j) = \delta_{ij}, \quad i, j = 1, \dots, k . \quad (11)$$

A(1): We choose $v_1 \in L_1 \setminus \{0\}$. Then there exists $\hat{w}_1 \in \mathbb{R}^{2n}$ with

$$\omega(v_1, \hat{w}_1) = 1 . \quad (12)$$

We set

$$w_1 := \Pi_2(\hat{w}_1) \quad (13)$$

and compute using the fact that L_1 is Lagrangian that

$$1 = \omega(v_1, \hat{w}_1) = \omega(v_1, \underbrace{\Pi_1(\hat{w}_1)}_{\in L_1}) + \omega(v_1, \underbrace{\Pi_2(\hat{w}_1)}_{=w_1}) = \omega(v_1, w_1) . \quad (14)$$

This verifies **A(1)**. Next we assume that $k \leq n - 1$ and prove **A(k)** \Rightarrow **A(k+1)**: Since

$$\dim \left(\bigcap_{i=1}^k \ker \omega(\cdot, w_i) \cap L_1 \right) \geq 2n - k - n \geq 1 \quad (15)$$

there exists $v_{k+1} \in L_1 \setminus \{0\}$ with

$$\omega(v_{k+1}, w_i) = 0, \quad 1 \leq i \leq k. \quad (16)$$

To show that v_1, \dots, v_{k+1} are linearly independent we assume that

$$\sum_{i=1}^{k+1} a_i v_i = 0 \quad (17)$$

and compute $j = 1, \dots, k$ using the induction hypothesis

$$0 = \omega\left(\sum_{i=1}^{k+1} a_i v_i, w_j\right) = \sum_{i=1}^{k+1} a_i \underbrace{\omega(v_i, w_j)}_{\delta_{ij}} = a_j. \quad (18)$$

Therefore, equation (17) is reduced to $a_{j+1} v_{j+1} = 0$ and we conclude $a_1 = \dots = a_{j+1} = 0$, proving linear independence. Since ω is non-degenerate there exists \hat{w}_{k+1} with

$$\omega(v_{k+1}, \hat{w}_{k+1}) = 1. \quad (19)$$

We set

$$w_{k+1} := \Pi_2(\hat{w}_{k+1}) \quad (20)$$

and conclude as in (14) that $\omega(v_{k+1}, w_{k+1}) = 1$. Proving that w_1, \dots, w_{k+1} are linearly independent is done exactly the same way as for v_1, \dots, v_{k+1} . This finishes the proof. \square

Now we prove Theorem 1.

Proof of Theorem 1. We consider the map

$$\begin{aligned} \mathrm{Sp}^R(n) &\rightarrow \mathcal{A}(n) \\ \psi &\mapsto R\psi \end{aligned} \quad (21)$$

which is well-defined since $R\psi$ is anti-symplectic and

$$(R\psi)^2 = R\psi R\psi = \psi^{-1}\psi = \mathbb{1}. \quad (22)$$

Moreover, the map is a diffeomorphism with inverse

$$\begin{aligned} \mathcal{A}(n) &\rightarrow \mathrm{Sp}^R(n) \\ S &\mapsto RS \end{aligned} \quad (23)$$

which again is well-defined since RS is symplectic and

$$R(RS)^{-1}R = RSRR = RS. \quad (24)$$

Thus, $\mathrm{Sp}^R(n) \cong \mathcal{A}(n)$. It remains to prove $\mathcal{A}(n) \cong \mathrm{Gl}(n, \mathbb{R}) \setminus \mathrm{Sp}(n)$. By Lemma 5 the conjugation map

$$\begin{aligned} \mathrm{Sp}(n) &\rightarrow \mathcal{A}(n) \\ \psi &\mapsto \psi^{-1}R\psi \end{aligned} \quad (25)$$

descends to a map

$$\mathfrak{C} : \mathrm{Gl}(n, \mathbb{R}) \setminus \mathrm{Sp}(n) \rightarrow \mathcal{A}(n). \quad (26)$$

In order to check that this is an isomorphism we observe that any involution S is diagonalizable with eigenvalues ± 1 . Indeed, if we set

$$V_{\pm 1} := \{v \in \mathbb{R}^{2n} \mid Sv = \pm v\} \quad (27)$$

then

$$\begin{aligned}\mathbb{R}^{2n} &\cong V_1 \oplus V_{-1} \\ v &\mapsto \frac{1}{2}(v + Sv) + \frac{1}{2}(v - Sv) .\end{aligned}\tag{28}$$

S being anti-symplectic implies that the linear spaces $V_{\pm 1}$ are isotropic since

$$\omega(v, w) = -\omega(Sv, Sw) = -\omega(v, w)\tag{29}$$

if $v, w \in V_1$ or $v, w \in V_{-1}$. Thus, by (28), $V_{\pm 1}$ are Lagrangian. Now we choose according to Lemma 6 a symplectic basis $v_1, \dots, v_n, w_1, \dots, w_n$ of \mathbb{R}^{2n} with v_1, \dots, v_n being a basis of L_1 and w_1, \dots, w_n being a basis of L_2 . We set

$$\psi^{-1} := (v_1, \dots, v_n, w_1, \dots, w_n) \in \mathrm{Sp}(n) .\tag{30}$$

With this definition it follows that

$$\psi^{-1}R\psi = S\tag{31}$$

and therefore the map \mathfrak{C} is surjective. To check injectivity we note that

$$\psi_1^{-1}R\psi_1 = \psi_2^{-1}R\psi_2 \iff \psi_2\psi_1^{-1}R\psi_1\psi_2^{-1} = R .\tag{32}$$

Thus by Lemma 5 the latter implies that $\psi_1\psi_2^{-1} \in \mathrm{Gl}(n, R)$. \square

Proof of Theorem 2. According to Wonenburger [Won66, Theorem 2] we can write any $\phi \in \mathrm{Sp}(n)$ as a product of two linear anti-symplectic involutions:

$$\phi = TS, \quad S, T \in \mathcal{A}(n) .\tag{33}$$

Using Lemma 6 as in the proof of Theorem 1 we can find a symplectic matrix $\psi \in \mathrm{Sp}(n)$ s.t.

$$\psi^{-1}R\psi = S .\tag{34}$$

If we set

$$\tilde{\phi} := \psi\phi\psi^{-1} = \psi T \psi^{-1} \psi S \psi^{-1} = \underbrace{\psi T \psi^{-1}}_{\tilde{T} \in \mathcal{A}(n)} R = \tilde{T} R\tag{35}$$

then we conclude

$$R\tilde{\phi}R = R\tilde{T}RR = R\tilde{T} = \tilde{\phi}^{-1}\tag{36}$$

i.e.

$$\tilde{\phi} \in \mathrm{Sp}^R(n) .\tag{37}$$

\square

3. CONCLUDING REMARKS

Remark 7. It is well-know that the homogeneous space $U(n)/O(n)$ is diffeomorphic to $U(n) \cap \mathrm{Sym}(n)$, where $\mathrm{Sym}(n)$ is the space of symmetric matrices in $\mathrm{Gl}(n, \mathbb{C})$:

$$U(n)/O(n) \cong \{\theta \in U(n) \mid \theta = \theta^T\} .\tag{38}$$

This can be seen similarly as in the proof of Theorem 1. The same maps restricted to the space of *orthogonal* anti-symplectic involutions give the above claimed identification.

We recall that $U(n)/O(n)$ is diffeomorphic to the Lagrangian Grassmannian $\mathcal{L}(n)$, that is the space of all linear Lagrangian subspaces of \mathbb{R}^{2n} , see [MS98, Lemma 2.31]

We close with the following unexpected observation, see Corollary 9. We point out that the space $\mathcal{A}(n)$ can be identified with the space of ordered Lagrangian splittings of \mathbb{R}^{2n} , cp. equation (28). Moreover, $\mathcal{A}(n)$ has a natural projection $\pi : \mathcal{A}(n) \rightarrow \mathcal{L}(n)$ to the Lagrangian Grassmannian $\mathcal{L}(n)$ given by $\pi(S) := \text{Fix}(S)$. The fiber $\pi^{-1}(L)$ consists of all Lagrangian splittings of the form $L \oplus \tilde{L}$. [MS98, Lemmas 2.30, 2.31] imply that every such \tilde{L} is the graph over L^\perp of a symmetric matrix. Thus, we can identify $\pi^{-1}(L) \cong \text{Sym}(n)$. Similarly the tangent space $T_L \mathcal{L}(n) \cong \text{Sym}(n)$. We proved

Lemma 8. *The space of linear anti-symplectic involutions is diffeomorphic to the tangent bundle of the Lagrangian Grassmannian*

$$\mathcal{A}(n) \cong T\mathcal{L}(n). \quad (39)$$

Corollary 9. *The tangent bundle of the Lagrangian Grassmannian can be given the structure of a homogenous space in two different ways.*

PROOF. One homogeneous structure can be obtained from the diffeomorphism $\mathcal{A}(n) \cong \text{Gl}(n, \mathbb{R}) \backslash \text{Sp}(n)$, see Theorem 1. The other from the Theorem in [BS72] based on semi-direct products and is diffeomorphic to $TU(n)/TO(n)$. \square

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REFERENCES

- [Bir15] G. D. Birkhoff, *The restricted problem of three bodies.*, Rend. Circ. Mat. Palermo **39** (1915), 265–334.
- [BS72] R. W. Brockett and H. J. Sussmann, *Tangent bundles of homogeneous spaces are homogeneous spaces*, Proc. Amer. Math. Soc. **35** (1972), 550–551.
- [EGH00] Y. Eliashberg, A. Givental, and H. Hofer, *Introduction to symplectic field theory*, Geom. Funct. Anal. (2000), no. Special Volume, Part II, 560–673, GAFA 2000 (Tel Aviv, 1999).
- [FvK12] U. Frauenfelder and O. van Koert, *The Hörmander index of symmetric periodic orbits*, 2012, arXiv:1208.4756.
- [MS98] D. McDuff and D. A. Salamon, *Introduction to symplectic topology*, second ed., Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 1998.
- [Won66] M. J. Wonenburger, *Transformations which are products of two involutions*, J. Math. Mech. **16** (1966), 327–338.

PETER ALBERS, MATHEMATISCHES INSTITUT, WESTFÄLISCHE WILHELMS-UNIVERSITÄT MÜNSTER
E-mail address: peter.albers@wwu.de

URS FRAUENFELDER, DEPARTMENT OF MATHEMATICS AND RESEARCH INSTITUTE OF MATHEMATICS,
SEOUL NATIONAL UNIVERSITY
E-mail address: frauenf@snu.ac.kr